MARKOFF–ROSENBERGER TRIPLES IN GEOMETRIC PROGRESSION

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Abstract. Solutions of the Markoff–Rosenberger equation $ax^2+by^2+cz^2 = dxyz$ such that their coordinates belong to the ring of integers of a number field and form a geometric progression are studied.

1. Introduction

The study of sequences of points on algebraic varieties is nowadays a topic in number theory studied for many authors. A special interest has been shown for the case of arithmetic progressions on plane curves. Let $C : F(x, y) = 0$ be a plane curve defined over a field K. An arithmetic progression (a.p. in what follows) of length n on C is a sequence of points $(x_1, y_1), \dots, (x_n, y_n) \in C(K)$ such that $x_1, \dots, x_n$ form an arithmetic progression. If C is an elliptic curve, many researchers have studied this problem depending on how C is described: Weierstrass form [10, 13, 15, 16, 6, 29], Mordell [22, 18], Congruent [11, 31], quartic form [32, 19, 5], Edwards [23], Huff [24]. If C has genus greater that 1 the case of hyperelliptic curve has been treated in [33, 4, 34]. As for the case of genus 0 is concerned: Pellian equations [14, 25, 1]; Conic section [7].

Recently, some authors have considered similar problems but replacing the arithmetic by geometric progressions (g.p. in what follows): Bérczes and Ziegler [8] and Bremner and Ulas [12] on Pell type equations.

Another point of view is consider a hypersurface $S : F(x_1, \dots, x_n) = 0$ in $\mathbb{A}^n$ and to study if the coordinates of a point in S, considered as a sequence, satisfy some property. For example, if they form an a.p.. Recently, the author and J. M. Tornero [17] have studied the case of triples in a.p. on the Markoff–Rosenberger equation over number field. That is, if $\mathcal{O}_K$ denotes the ring of integers of a number field K, the paper [17] studies triples $x, y, z \in \mathcal{O}_K$ in a.p. such that $ax^2 + by^2 + cz^2 = dxyz$ for some $a, b, c, d \in \mathcal{O}_K$. In this paper, we fix our attention in the case of g.p. instead of a.p. and our intention is to prove analogous results for g.p. as obtained in [17]. That is, our main objective is to study the set

$$\mathcal{GP}_{(a,b,c,d)}(K) := \{\mathcal{O}_K\text{–non-trivial triples in g.p. to } ax^2 + by^2 + cz^2 = dxyz\}.$$

In section 2, we will see that $\mathcal{GP}_{(a,b,c,d)}(K)$ is in bijection with a subset of the affine solutions over $\mathcal{O}_K$ of a genus zero curve, denoted by $\mathcal{G}_{(a,b,c,d)}$. Then the description of $\mathcal{GP}_{(a,b,c,d)}(K)$ is translated to the computation of the solutions of a genus zero curve. For this task we will use mainly and heavily the research of Poulakis on solving genus zero Diophantine equations: the papers jointly written with Alvanos, Bilu and Voskos [2, 3, 27, 26]. In section 3, we present an algorithm based on Alvanos and Poulakis' work [3] that allows us to compute $\mathcal{G}_{(a,b,c,d)}(\mathcal{O}_K)$. We show how the algorithm described in this section works on some examples not covered by the theoretical results from section 4.

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All the main theoretical results appear at section 4. There, we show that infinitely many integral Markoff–Rosenberger triples in g.p. over a number field K can exist only if K neither is the rational field nor a quadratic imaginary field. When K is the rational field we give an explicit description of this finite set. Finally we fix our attention to our original goal, the study of the generalized Markoff equation: $x^2+y^2+z^2=dxyz$ where $d\in\mathbb{Z}_{>0}$. We obtain the finite set $\mathcal{GP}_{(1,1,1,d)}(K)$ when K is the rational field or an imaginary quadratic field. Furthermore, for the case whether K is a real quadratic field we give a completely explicit description of this set, that could be either empty or infinite.

2. A genus zero curve

Our starting point is the Markoff equation

$$x^2+y^2+z^2=3xyz. \tag{1}$$

The integer solutions (so–called Markoff triples) of this Diophantine equation were deeply studied by Markoff in [20, 21] obtaining, among other results, that infinitely many Markoff triples exist. Later on, the Markoff equation has been generalized for several authors. We are going to focus our attention to the one studied by Rosenberger [28]:

$$ax^2+by^2+cz^2=dxyz. \tag{2}$$

We will call this equation the Markoff–Rosenberger equation and its solution will be called Markoff–Rosenberger triples. Notice that Rosenberger imposed some extra conditions on the coefficients $a,b,c,d\in\mathbb{N}$: $a|d$, $b|d$, $c|d$ and $(a,b)=(a,c)=(b,c)=1$. With these requirements he proved that non–trivial integral Markoff–Rosenberger triples exist only if (a,b,c,d) belongs to the set $\{(1,1,1,1), (1,1,1,3), (1,1,2,2), (1,1,2,4), (1,2,3,6), (1,1,5,5)\}$. We will not assume these extra conditions for the rest of the paper.

This paper is devoted to the study of Markoff–Rosenberger triples that form a g.p. over $\mathcal{O}_K$. Let be $a,b,c,d\in\mathcal{O}_K$ and $x,y,z\in\mathcal{O}_K$ a Markoff–Rosenberger triples in g.p.. Then $\alpha,\beta\in\mathcal{O}_K$ exist such that

$$x=\alpha,\quad y=\alpha\beta,\quad z=\alpha\beta^2,$$

satisfying

$$\alpha^2(c\beta^4-d\alpha\beta^3+b\beta^2+a)=0.$$

Therefore, if we exclude the solutions with $\alpha=0$ that correspond to the trivial solution $(x,y,z)=(0,0,0)$, we obtain that non-trivial Markoff–Rosenberger triples in g.p. over $\mathcal{O}_K$ are in bijection with the affine solutions over $\mathcal{O}_K$ of the curve

$$\mathcal{G}=\mathcal{G}_{(a,b,c,d)}\,:\,F(X,Y)=cY^4-dXY^3+bY^2+a=0.$$

with $X\neq 0$. Namely:

$$\begin{array}{ccc} \mathcal{G}_{(a,b,c,d)}(\mathcal{O}_K) & \longrightarrow & \mathcal{GP}_{(a,b,c,d)}(K) \\ (X,Y) & \longmapsto & (X,XY,XY^2) \\ (x,z/y) & \longleftarrow & (x,y,z) \end{array} \tag{3}$$

Let $\widetilde{F}(X,Y,Z)$ be the homogenization of $F(X,Y)$ and denote by $\widetilde{\mathcal{G}}$ the projective curve defined by $\widetilde{\mathcal{G}}\,:\,\widetilde{F}(X,Y,Z)=0$. Now, the curve $\widetilde{\mathcal{G}}$ has two points at infinity: $[c:d:0]$ and the singular point $[1:0:0]$. In particular, $\widetilde{\mathcal{G}}$ has genus 0 and the birational map

$$\begin{array}{cccc} \widetilde{\psi}: & \mathbb{P}^1 & \longrightarrow & \widetilde{\mathcal{G}} \\ & [U:V] & \longmapsto & [cU^4+bV^2U^2+aV^4\,:\,dU^4\,:\,dU^3V] \end{array} \tag{4}$$

gives a parametrization of the curve $\widetilde{\mathcal{G}}$ over K.

Denote by $\widetilde{\mathcal{G}}_\infty$ the set of infinity places of the field $\overline{K}(C)$. Then by [26, Lemma 2.2] we have that

$$|\widetilde{\mathcal{G}}_\infty| = |\{[U:V] \in \mathbb{P}^1 \,:\, dU^3V = 0\}| = |\{[1:0],[0:1]\}| = 2.$$

Note that both points at infinity are defined over $\mathbb{Q}$.

3. The general algorithm over number fields

Let K be a number field of degree $n = [K:\mathbb{Q}]$ and $a,b,c,d \in \mathcal{O}_K$. Our objective in this section is to describe the set

$$\mathcal{G}(\mathcal{O}_K) = \{(X,Y) \in {\mathcal{O}_K}^2 \,|\, F(X,Y) = 0\}.$$

For this purpose we use the affine part of the parametrization $\widetilde{\psi}$:

$$\begin{array}{rccc} \psi: & \mathbb{P}^1 & \longrightarrow & \mathcal{G} \\ & [U:V] & \longmapsto & \left(\dfrac{cU^4+bV^2U^2+aV^4}{dU^3V}, \dfrac{U}{V}\right) \end{array} \tag{5}$$

and we are going to develop an algorithm heavily based on the Alvanos and Poulakis' algorithm `INTEGRAL-POINTS2A` from the paper [3]. In fact, the algorithm presented here is just the application of `INTEGRAL-POINTS2A` to the curve $\mathcal{G}$.

Step 1: Let $\alpha \in \mathcal{O}_K$ and denote by $\widehat{\alpha} = \alpha^s$ where $s = 0$ or $s = 1$ depending on whether $\alpha \in \mathbb{Z}$ or $\alpha \notin \mathbb{Z}$. Let $E = \mathbb{Q}(\widehat{c})$ and $L = \mathbb{Q}(\widehat{d})$. Compute

$$\delta_0 = gcd\left(\frac{c}{\widehat{c}}\mathcal{N}_E(\widehat{c}), \frac{d}{\widehat{d}}\mathcal{N}_L(\widehat{d})\right) \in \mathbb{Z}.$$

We denote by $\mathcal{N}_K$ the absolute norm map for a number field K.

Step 2: Compute

$$M = \left\{t \in \mathcal{O}_K \,|\, \mathcal{N}_K(t) \text{ divides } \mathcal{N}_K(a)\delta_0^{4n}\right\}_{/\sim},$$

where $\sim$ denotes equivalence class (modulo associated elements).

Step 3: Compute a basis for the unit group $\mathcal{U}(\mathcal{O}_K)$. By Dirichlet's Unit Theorem we have that $\mathcal{U}(\mathcal{O}_K)$ is a finitely generated abelian group and therefore:

$$\mathcal{U}(\mathcal{O}_K) = \langle \zeta_k \rangle \oplus \langle \varepsilon_1, \ldots, \varepsilon_r \rangle$$

where ζ_k is a k-th root of unity (k is the torsion order) and r is the rank. Moreover, $r = r_1 + r_2 - 1$ where r_1 denotes the number of real embeddings of K and r_2 the number of complex pairs embeddings of K. In particular, $\mathcal{U}(\mathcal{O}_K)$ is finite if and only if $K = \mathbb{Q}$ or K is an imaginary quadratic field.

Step 4: For any $t \in M$ compute $\tau(i,t)$ the order of the class of ε_i in $\mathcal{U}(\mathcal{O}_K/\delta_0 dt^3)$. Note that if $\delta_0 dt^3 \in \mathcal{O}_K$ then we define $\tau(i,t) = 1$.

Step 5: Now, for every $t \in M$ compute the set $H(t)$ of units $\eta = \zeta_k^l \varepsilon_1^{l_1} \cdots \varepsilon_r^{l_r}$ with $0 \le l < k, 0 \le l_i < \tau(i,t)$ for $i = 1, \ldots, r$ such that $\psi(t\eta, \delta_0) \in \mathcal{O}_K^2$. From this condition we obtain that $d(t\eta)^3\delta_0$ divides $c(t\eta)^4 + b\delta_0^2(t\eta)^2 + a\delta_0^4$ and δ_0 divides $t\eta$. Then

$$H(t) = \left\{ \eta = \zeta_k^l \varepsilon_1^{l_1} \cdots \varepsilon_r^{l_r} \;\middle|\; \begin{array}{c} 0 \le l < k, \quad 0 \le l_i < \tau(i,t), \;\; i = 1, \ldots, r \\ d(t\eta)^3\delta_0 | (c(t\eta)^4 + b\delta_0^2(t\eta)^2 + a\delta_0^4) \text{ and } \delta_0 | t\eta \end{array} \right\}.$$

Notice that in the case that $\delta_0 \neq \pm 1$ we have δ_0 divides t.

OUTPUT: Denote by $\Theta(t) = \left\{ \prod_{i=1}^r \varepsilon_i^{\tau(i,t)z_i} \;\middle|\; z_i \in \mathbb{Z}, i = 1, \ldots, r \right\}$. Then

$$\mathcal{G}(\mathcal{O}_K) = \bigcup_{t \in M} \{\psi(t\eta\varepsilon, \delta_0) \,|\, \eta \in H(t) \text{ and } \varepsilon \in \Theta(t)\}.$$

Note that in order to apply the algorithm for some fixed values a,b,c,d in the ring of integers of some number field K we should be able to solve some problems.

All of them are sorted out in `Magma` [9]. In the following table we show the main problem to be solved and the `Magma` functions that may be used:

Step	Problems	`Magma` functions
2	M	`NormEquation`
3	$\mathcal{U}(\mathcal{O}_K)$	`UnitGroup`
4	$\tau(i,t)$	`quo`, `MultiplicativeGroup`, `Order`
5	$\psi(t\eta,\delta_0)\in\mathcal{O}_K^2$	`IsIntegral`

3.1. **The algorithm at work.** We recall that Rosenberger [28] proved that integral Markoff–Rosenberger triples exist if and only if

$$(a,b,c,d)\in\{(1,1,1,1),\ (1,1,1,3),\ (1,1,2,2),\ (1,1,2,4),\ (1,2,3,6),\ (1,1,5,5)\},$$

with the extra requirements: $a,b,c,d\in\mathbb{N}$: $a|d$, $b|d$, $c|d$ and $(a,b)=(a,c)=(b,c)=1$. Note that Corollary 4.4 applies to the previous cases tell us that

(a,b,c,d)	$\mathcal{GP}_{(a,b,c,d)}(\mathbb{Q})$
$(1,1,1,1)$	$(\pm 3,3,\pm 3)$
$(1,1,1,3)$	$(\pm 1,1,\pm 1)$
$(1,1,2,2)$	$(\pm 2,2,\pm 2)$
$(1,1,2,4)$	$(\pm 1,1,\pm 1)$
$(1,2,3,6)$	$(\pm 1,1,\pm 1)$
$(1,1,5,5)$	$\emptyset$

That is, integral Markoff–Rosenberger triples in g.p. exist for all these cases except for the last one. In this section we are going to study the case $(1,1,5,5)$ over the first two quadratic real fields. Thanks to Theorem 4.1 we know that if D is a squarefree positive integer then a Markoff–Rosenberger triple in g.p. over $\mathbb{Q}(\sqrt{D})$ exists if and only if infinitely many exist. We show one case of each of these possibilities. In particular, in $\mathbb{Q}(\sqrt{2})$ we describe the infinitely many triples in g.p.; meanwhile in $\mathbb{Q}(\sqrt{3})$ we will show that triples in g.p. do not exist. Let us apply our algorithm for this purpose.

Let K be a real quadratic field. We have that $\delta_0=5$, in particular $M=\left\{t\in\mathcal{O}_K\,|\,\mathcal{N}_K(t)=\pm 5^k,\ k=2,\dots,8\right\}_{/\sim}$. Moreover, a fundamental unit ε_D exist, such that $\mathcal{U}(\mathcal{O}_K)=\{\pm\varepsilon_D^k\,|\,k\in\mathbb{Z}\}$. Now we work out the cases $K=\mathbb{Q}(\sqrt{2})$ and $K=\mathbb{Q}(\sqrt{3})$:

• $K=\mathbb{Q}(\sqrt{2})$. The fundamental unit is $\varepsilon_2=1+\sqrt{2}$ which has order 12 on $\mathcal{U}(\mathcal{O}_K/5)$. We have $M=\bigcup_{k=1}^{4}\{5^k,5^k\varepsilon_2\}$. Then, for any $t\in M$ we obtain that $H(t)=\emptyset$, except in the following two cases:

$$H(5\varepsilon_2)=\{\pm\varepsilon_2^2,\pm\varepsilon_2^8\}\quad\text{and}\quad H(5)=\{\pm\varepsilon_2^3,\pm\varepsilon_2^9\}.$$

Now, we have $\Theta(t)=\{\varepsilon_2^{12z}\,|\,z\in\mathbb{Z}\}$ for any $t\in M$. Therefore

$$\mathcal{G}_{(1,1,5,5)}(\mathcal{O}_{\mathbb{Q}(\sqrt{2})})=\bigcup_{k\in\{1,2\}}\bigcup_{z\in\mathbb{Z}}\{\psi(\pm 5\varepsilon_2^{3^k+12z},5)\}.$$

Then we have obtained infinitely many triples in g.p., describe by

$$\mathcal{GP}_{(1,1,5,5)}(\mathbb{Q}(\sqrt{2}))=\bigcup_{k=1}^{2}\bigcup_{z\in\mathbb{Z}}\left\{(\pm\beta,\beta\gamma,\pm\beta\gamma^2)\ \middle|\ \begin{array}{l}\gamma=\varepsilon_2^{3^k+12z},\\ \beta=\gamma+5^{-1}(\gamma^{-1}+\gamma^{-3})\end{array}\right\}.$$

• $K=\mathbb{Q}(\sqrt{3})$. The fundamental unit is $\varepsilon_3=2+\sqrt{3}$ which has order 3 on $\mathcal{U}(\mathcal{O}_K/5)$. In this case, we have $M=\{5^k\,|\,k=1,2,3,4\}$. Finally, we obtain that $H(t)=\emptyset$ for all $t\in M$. That is, $\mathcal{G}_{(1,1,5,5)}(\mathcal{O}_{\mathbb{Q}(\sqrt{3})})=\emptyset$, and therefore

$$\mathcal{GP}_{(1,1,1,5)}(\mathbb{Q}(\sqrt{3}))=\emptyset.$$

4. Theoretical results

This section is dedicated to show the theoretical results obtained on Markoff–Rosenberger triples in g.p. over number fields.

Theorem 4.1. *Let K be a number field and $a,b,c,d \in \mathcal{O}_K$. Assume that $\mathcal{G}(\mathcal{O}_K)$ contains a non–singular point, then the set $\mathcal{GP}_{(a,b,c,d)}(K)$ is finite if and only if $K=\mathbb{Q}$ or K is a imaginary quadratic field.*

Proof. Let C be an affine algebraic curve of genus g defined over a number field K and denote by C_∞ the set of infinity places of the field $\overline{K}(C)$. Siegel [30] proved that if $g>0$ or $|C_\infty|>2$ then $C(\mathcal{O}_K)$ is finite. However, $C(\mathcal{O}_K)$ may be finite if $g=0$ and $|C_\infty|\le 2$. These last cases where treated by Alvanos, Bilu and Poulakis [2], obtaining a complete characterization on the finiteness of $C(\mathcal{O}_K)$. In particular, our genus zero curve $\mathcal{G}$ satisfies $|\mathcal{G}_\infty|=2$ and both points at infinity are defined over $\mathbb{Q}$, therefore [2, Theorem 1.2] asserts that $\mathcal{G}(\mathcal{O}_K)$ is finite if and only if $K=\mathbb{Q}$ or K is a imaginary quadratic field. $\square$

Remark 4.2. Assume that d divides $a+b+c$ on $\mathcal{O}_K$ where K neither is $\mathbb{Q}$ nor a imaginary quadratic field, then $\#\mathcal{GP}_{(a,b,c,d)}(K)=\infty$.

Now we fix our attention on the case of the rational field or a imaginary quadratic field, since in this cases we have proved that only a finite number of triples in g.p. exist. First the rational case:

Theorem 4.3. *Let $a,b,c,d\in\mathbb{Z}$. For any $u\in\mathbb{Z}$ denote by $\gamma_u=\frac{cu^4+bu^2+a}{du^3}$ then*

$$\mathcal{GP}_{(a,b,c,d)}(\mathbb{Q})=\left\{\left(\gamma_u,u\gamma_u,u^2\gamma_u\right)\,\middle|\,u|a,\ u>0,\ du^3|(cu^4+bu^2+a)\right\}.$$

Proof. We are going to apply the algorithm develoved by Poulakis and Voskos [27, §4] that allows to compute $C(\mathbb{Z})$ when C is a genus zero curve such that $|C_\infty|=2$. In our case we will apply it on the curve $\mathcal{G}_{(a,b,c,d)}$:
Step 1: There is only one singular point: $[1:0:0]$ and it is at "infinity". Therefore this point does not belong to the affine points of $\mathcal{G}_{(a,b,c,d)}$.
Step 2: From the parametrization $\widetilde{\psi}$, see equation (4), we have

$$p(U,V)=cU^4+bV^2U^2+aV^4,\quad q(U,V)=dU^4,\quad r(U,V)=dU^3V.$$

Therefore we are at the case (i).
Step 3: $b_0=b_1=b_2=b_3=a_1=a_3=0$, $b_4=d$,$a_0=a$, $a_4=c$ and $a_2=b$.
Step 4: $\delta_1=gcd(a,0)=a$ and $\delta_2=gcd(c,d)$.
Step 5: $\Sigma=\{(u,v)\in\mathbb{Z}^2\mid gcd(u,v)=1,\ u>0,\ u|a,\ v|gcd(c,d)\}$
Step 6: We compute for any $(u,v)\in\Sigma$

$$x(u,v)=\frac{p(u,v)}{r(u,v)}=\frac{cu^4+bv^2u^2+av^4}{du^3v}\quad\text{and}\quad y(u,v)=\frac{q(u,v)}{r(u,v)}=\frac{du^4}{du^3v}=\frac{u}{v}.$$

Now, $x(u,v)$ and $y(u,v)$ must belong to $\mathbb{Z}$. Therefore $v|u$, and since $gcd(u,v)=1$ we have that $v=\pm1$. Then the Poulakis and Voskos' algorithm outputs

$$\mathcal{G}_{(a,b,c,d)}(\mathbb{Z})=\left\{\pm\left(\frac{cu^4+bu^2+a}{du^3},u\right)\in\mathbb{Z}^2\,\middle|\,u|a,\ u>0,\ du^3|(cu^4+bu^2+a)\right\}.$$

Now, thanks to the bijection given by (3) we obtain the result. $\square$

Corollary 4.4. *Let $a,b,c,d\in\mathbb{Z}$ with a squarefree. Denote by $\gamma=\frac{a+b+c}{d}$. Then*

$$\mathcal{GP}_{(a,b,c,d)}(\mathbb{Q})=\begin{cases}\{(\pm\gamma,\gamma,\pm\gamma)\} & \text{if } d|(a+b+c),\\ \emptyset & \text{otherwise.}\end{cases}$$

Proof. Assume that $\gamma_u \in \mathbb{Z}$ with $u|a$. Let us prove that $u = \pm 1$. Now, since a is squarefree we have that $a = u \cdot w$ for some integer w coprime with u. Then

$$\gamma_u = \frac{cu^4 + bu^2 + a}{du^3} = \frac{cu^3 + bu + w}{du^2} \in \mathbb{Z}.$$

Therefore, $u|(cu^3 + bu + w)$. On the other hand, $u|(cu^3 + bu)$. Both conditions implies that $u|w$ which is equivalent to $u = \pm 1$ since $gcd(u, w) = 1$. Applying the theorem above we have that $\gamma_{\pm 1} = \pm\gamma$ and a necessary condition for $\gamma \in \mathbb{Z}$ is that $d|(a + b + c)$. □

Remark 4.5. Notice that the condition on the squarefreeness of a is necessary. For example, $\mathcal{GP}_{(4,1,1,1)}(\mathbb{Q}) = \{(\pm 6, 6, \pm 6), (\pm 3, 6, \pm 12)\}$.

The next two results give complete descriptions on the case of generalized Markoff triples in g.p. over number fields of very low degree, that is one or two. Notice that if (x, y, z) is a triple in g.p. to the equation $x^2 + y^2 + z^2 = dxyz$ if and only if $(x, -y, z)$ is a triple in g.p. to the equation $x^2 + y^2 + z^2 = -dxyz$. Then we will assume that d is positive. Now, the first result describes precisely when only finitely many triples in g.p. exist.

Theorem 4.6. *Let $d, D \in \mathbb{Z}_{>0}$, with D squarefree. If $(d, D) \neq (1, 1)$, then*

$$\mathcal{GP}_{(1,1,1,d)}(\mathbb{Q}(\sqrt{-D})) = \mathcal{GP}_{(1,1,1,d)}(\mathbb{Q}) = \begin{cases} \{(\pm 3, 3, \pm 3)\} & \text{if } d = 1, \\ \{(\pm 1, 1, \pm 1)\} & \text{if } d = 3, \\ \emptyset & \text{if } d \neq 1, 3, \end{cases}$$

and $\mathcal{GP}_{(1,1,1,1)}(\mathbb{Q}(i)) = \{(\pm 3, 3, \pm 3), (\pm i, -1, \mp i)\}$.

Proof. First notice that the second equality, for $(d, D) \neq (1, 1)$, is a consequence of Corollary 4.4 with $a = b = c = 1$. To prove the first equality we use the algorithm of section 3. We have that $\delta_0 = 1$. Let $K = \mathbb{Q}(\sqrt{-D})$ then $M = \{1\}$, since on an imaginary quadratic field all elements have positive norm. Now we should compute the unit group of $\mathcal{O}_K$. But it is well–known that $\mathcal{U}(\mathcal{O}_K) = \langle \zeta_k \rangle$, when $k = 2$ ($\zeta_2 = -1$) if $D \neq 1, 3$, $k = 4$ ($\zeta_4 = i$) if $D = 1$ and $k = 6$ ($\zeta_6 = (1 + \sqrt{-3})/2$) if $D = 3$. Next step is to compute the set $H(1)$: in this case its elements are ζ_k^l for $0 \leq l < k$ such that $d|(\zeta_k^{4l} + \zeta_k^{2l} + 1)$. It is a straightforward computation to determine $H(1)$ depending on D and d:

$$H(1) = \begin{cases} \emptyset & \text{if } D \neq 3 \text{ and } d \neq 1, 3, \\ \{\pm 1\} & \text{if } D \neq 1, 3 \text{ and } d = 1, 3, \\ \{\zeta_6^l \,|\, l = 1, 2, 4, 5\} & \text{if } D = 3 \text{ and } d \neq 1, 3, \\ \{\zeta_6^l \,|\, 0 \leq l < 6\} & \text{if } D = 3 \text{ and } d = 1, 3, \\ \{\zeta_4^l \,|\, 0 \leq l < 4\} & \text{if } D = 1 \text{ and } d = 1, \\ \{\pm 1\} & \text{if } D = 1 \text{ and } d = 3. \end{cases}$$

Then the algorithm outputs:

$$\mathcal{G}(\mathcal{O}_K) = \left\{ \left(\frac{\eta^4 + \eta^2 + 1}{d\eta^3}, \eta \right) \,\Big|\, \eta \in H(1) \right\}.$$

That is

$$\mathcal{G}(\mathcal{O}_K) = \begin{cases} \emptyset & \text{if } D \neq 3 \text{ and } d \neq 1, 3, \\ \{\pm(3, 1)\} & \text{if } D \neq 1, 3 \text{ and } d = 1, \\ \{\pm(1, 1)\} & \text{if } D \neq 3 \text{ and } d = 3, \\ \{(0, \zeta_6^l) \,|\, l = 1, 2, 4, 5\} & \text{if } D = 3 \text{ and } d \neq 1, 3, \\ \{(0, \zeta_6^l) \,|\, l = 1, 2, 4, 5\} \cup \{\pm(3, 1)\} & \text{if } D = 3 \text{ and } d = 1, \\ \{(0, \zeta_6^l) \,|\, l = 1, 2, 4, 5\} \cup \{\pm(1, 1)\} & \text{if } D = 3 \text{ and } d = 3, \\ \{\pm(3, 1), \pm(i, i)\} & \text{if } D = 1 \text{ and } d = 1. \end{cases}$$

Therefore, the bijection given by (3) gives:

$$\mathcal{GP}_{(1,1,1,d)}(\mathbb{Q}(\sqrt{-D})) = \begin{cases} \emptyset & \text{if } d \neq 1,3, \\ \{(\pm 1, 1, \pm 1)\} & \text{if } d = 3, \\ \{(\pm 3, 3, \pm 3)\} & \text{if } D \neq 1 \text{ and } d = 1, \\ \{(\pm 3, 3, \pm 3), (\pm i, -1, \mp i)\} & \text{if } D = 1 \text{ and } d = 1. \end{cases}$$

□

Remark 4.7. In particular the previous result proves:

$$\bigcup_{d,D\in\mathbb{Z}_{>0}} \mathcal{GP}_{(1,1,1,d)}(\mathbb{Q}(\sqrt{-D})) = \{(\pm 1, 1, \pm 1), (\pm 3, 3, \pm 3), , (\pm i, -1, \mp i)\}$$

Remark 4.8. Note that a similar study for fixed $a, b, c \in \mathbb{Z}$, with a squarefree, could be done. That is, to compute explicitly $\mathcal{GP}_{(a,b,c,d)}(\mathbb{Q}(\sqrt{-D}))$ and $\mathcal{GP}_{(a,b,c,d)}(\mathbb{Q})$ for any $d, D \in \mathbb{Z}_{>0}$, with D squarefree.

Once we have treated the cases where there are only a finite number of generalized Markoff triples in g.p., we are going to give a complete description of the set of generalized Markoff triples in g.p. over a real quadratic field. In this case, we will have infinitely many such triples or none at all.

Theorem 4.9. *Let D be a squarefree positive integer and ε_D be the fundamental unit of the real quadratic field $K = \mathbb{Q}(\sqrt{D})$. Denote by n the order of the class of ε_D in $\mathcal{U}(\mathcal{O}_K/d)$ and define the set*

$$\mathcal{H} = \left\{k \in \{0, \dots, n\} \mid \varepsilon_D^{4k} + \varepsilon_D^{2k} + 1 \equiv 0 \,(mod(d\,\mathcal{O}_K))\right\}.$$

Then

$$\mathcal{GP}_{(1,1,1,d)}(\mathbb{Q}(\sqrt{D})) = \bigcup_{k\in\mathcal{H}} \bigcup_{z\in\mathbb{Z}} \left\{ (\pm\beta, \beta\gamma, \pm\beta\gamma^2) \,\middle|\, \gamma = \varepsilon_D^{nz+k},\ \beta = \frac{1}{d}\sum_{j=0}^{2} \gamma^{1-2j} \right\}.$$

Proof. To prove this result we are going to use the algorithm of section 3. First of all, let ε_D the fundamental unit of the real quadratic field $K = \mathbb{Q}(\sqrt{D})$. That is, ε_D satisfies $\mathcal{U}(\mathcal{O}_K) = \{\pm\varepsilon_D^k \mid k \in \mathbb{Z}\}$. Then we have step 3. To compute M we have only to observe that since $\delta_0 = 1$ the set M consists on 1 and ε_D in the case that $\mathcal{N}_K(\varepsilon_D) = -1$. Note that this is what is so-called the negative Pell equation. While the Pell equation $\mathcal{N}_K(\varepsilon_D) = 1$ has always a solution for any positive integer D, the negative Pell equation does not always have one, although there exist an effective algorithm to determine, for a fixed D, which is the case.

Now, denote by n the order of the class of ε_D in $\mathcal{U}(\mathcal{O}_K/d)$. Then at step 4 we must compute $H(t)$ for $t \in M$. Note that 1 always belongs to M, then we have

$$H(1) = \left\{\pm\varepsilon_D^k \mid \varepsilon_D^{4k} + \varepsilon_D^{2k} + 1 \equiv 0 \,(\mathrm{mod}(d\mathcal{O}_K)),\ k = 0, \dots, n-1\right\}.$$

In the case that $\mathcal{N}_K(\varepsilon_D) = -1$ we have that $\varepsilon_D \in M$ and

$$H(\varepsilon_D) = \left\{\pm\varepsilon_D^k \mid \varepsilon_D^{4(k+1)} + \varepsilon_D^{2(k+1)} + 1 \equiv 0 \,(\mathrm{mod}(d\mathcal{O}_K)),\ k = 0, \dots, n-1\right\}.$$

Finally, we have $\Theta(t) = \left\{\varepsilon_D^{nz} \,\middle|\, z \in \mathbb{Z}\right\}$ for any $t \in M$. Then the algorithm outputs

$$\mathcal{G}_{(1,1,1,d)}(\mathcal{O}_K) = \{\psi(\pm\varepsilon_D^{k+nz}, 1) \mid k \in \mathcal{H},\ z \in \mathbb{Z}\}.$$

Then the bijection given by (3) jointly with the definition of the parametrization ψ given by (5) give the result. □

Remark 4.10. The triples in g.p. over the ring of integer of a real quadratic field of the generalized Markoff equation $x^2+y^2+z^2=dxyz$ have a very special shape. That is, let $(\beta,\gamma\beta,\gamma^2\beta)$ such a triple. Then $\gamma=\varepsilon_D^{nz+k}$ and $\beta=\frac{1}{d}\sum_{j=0}^{2}\gamma^{1-2j}$ for some $n,k,z\in\mathbb{Z}$, where ε_D is the fundamental unit on the real quadratic field $\mathbb{Q}(\sqrt{D})$. Moreover, $\gamma\beta\in\mathbb{Z}$ since $\gamma\beta=\frac{1}{d}(1+\mathcal{T}race_K(\gamma^2))\in\mathcal{O}_K$ and the trace of any $\alpha\in\mathcal{O}_K$ belongs to $\mathbb{Z}$. But there is something more. The third term of the g.p. satisfies $\gamma^2\beta=\mathcal{N}_K(\gamma)^\sigma\beta=\pm^\sigma\beta$ where σ is the nontrivial automorphism of the Galois group of $\mathbb{Q}(\sqrt{D})$ and the sign depends on $\mathcal{N}_K(\varepsilon_D)$ and on $n,k,z\in\mathbb{Z}$.

Example 4.11. *We are looking for all Markoff triples in g.p. over $\mathbb{Q}(\sqrt{5})$. First, the fundamental unit is $\varepsilon_5=(1+\sqrt{5})/2$ whose class on $\mathcal{U}(\mathcal{O}_{\mathbb{Q}(\sqrt{5})}/3)$ has order $n=8$. Next, we obtain $\mathcal{H}=\{0,4,8\}$. Therefore, we conclude:*

$$\mathcal{G}_{(1,1,1,3)}(\mathbb{Q}(\sqrt{5}))=\bigcup_{z\in\mathbb{Z}}\left\{(\pm\beta,\beta\gamma,\pm\beta\gamma^2)\;\middle|\;\gamma=\varepsilon_5^{4z},\ \beta=\frac{1}{3}\sum_{j=0}^{2}\gamma^{1-2j}\right\}.$$

The following table shows some examples of Markoff triples in g.p. over $\mathbb{Q}(\sqrt{5})$:

z	*Markoff triples in g.p. over* $\mathbb{Q}(\sqrt{5})$
0	$(1,1,1)$
1	$(56-24\sqrt{5},16,56+24\sqrt{5})$
2	$(17296-7728\sqrt{5},736,17296+7728\sqrt{5})$
3	$(5564321-2488392\sqrt{5},34561,5564321+2488392\sqrt{5})$
4	$(1791660256-801254496\sqrt{5},1623616,1791660256+801254496\sqrt{5})$

Finally, we present a partial result valid for any number field.

Theorem 4.12. *Let K be a number field and $a,b,c,d\in\mathcal{O}_K$ such that $a,d\in\mathcal{U}(\mathcal{O}_K)$. For any $u\in\mathcal{U}(\mathcal{O}_K)$ denote by $\gamma_u=d^{-1}(cu+bu^{-1}+au^{-3})$. Then*

$$\mathcal{GP}(a,b,c,d,K)=\{(\gamma_u,u\gamma_u,u^2\gamma_u)\,|\,u\in\mathcal{U}(\mathcal{O}_K)\}.$$

Proof. Let us apply the algorithm of section 3. Now, since $a,d\in\mathcal{U}(\mathcal{O}_K)$ we have $\delta_0=1$, $M=\{t\in\mathcal{O}_K\,|\,\mathcal{N}_K(t)=\pm1\}_{/\sim}$ and $\tau(i,t)=1$ for $i=1,\dots,r=\mathrm{rank}_{\mathbb{Z}}\mathcal{U}(\mathcal{O}_K)$ and any $t\in M$. Therefore, if $t\in M$ we have that $H(t)=\mathcal{U}(\mathcal{O}_K)_{tors}$ and $\Theta(t)=\mathcal{U}(\mathcal{O}_K)_{free}$, the torsion and free part of the unit group $\mathcal{U}(\mathcal{O}_K)$ respectively. Then the algorithm outputs

$$\mathcal{G}(\mathcal{O}_K)=\bigcup_{t\in M}\{\psi(t\eta\varepsilon,1)\,|\,\eta\in\mathcal{U}(\mathcal{O}_K)_{tors}\,,\ \varepsilon\in\mathcal{U}(\mathcal{O}_K)_{free}\}=\bigcup_{u\in\mathcal{U}(\mathcal{O}_K)}\{\psi(u,1)\}.$$

Then we conclude the proof using (3) and (5). □

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